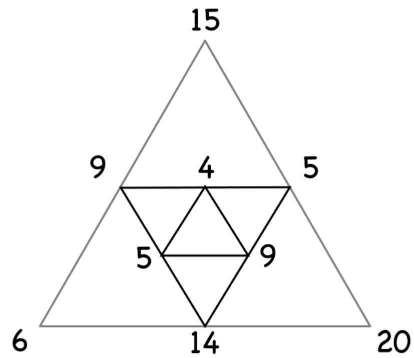
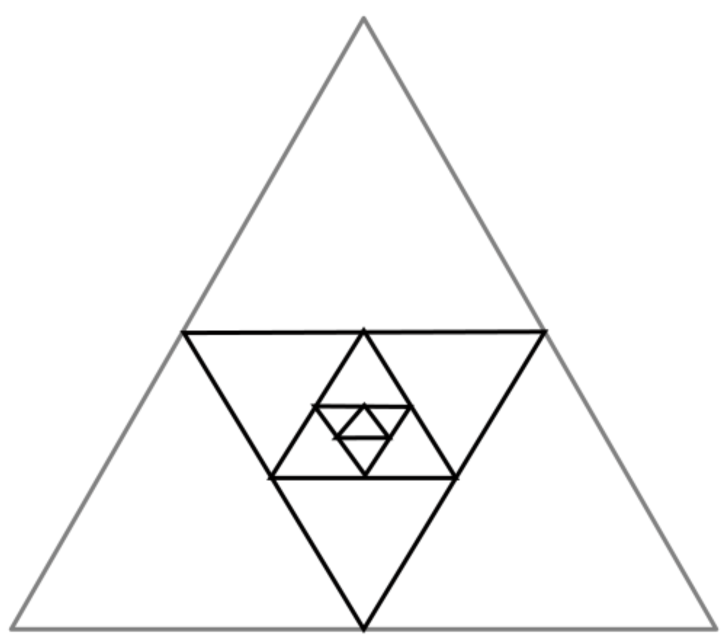
Puzzle of the Week

Difference Triangles

Start by putting numbers in the corners of a triangle. Then, put the differences of those numbers in the middle of the sides and make a new triangle with those differences. Continue this process and see where it takes you.



THE CHALLENGE: 1) Are there starting numbers for which this will go on forever without repeating, or does it always settle down into a simple pattern eventually? 2) If there is a simple ending pattern, what is it? 3) For which starting numbers will it end up with three 0's?



Puzzle of the Week
Difference Triangles – Notes

THE CHALLENGE: First, just play with it and see what happens from various groups of starting numbers. Look for patterns and enjoy the experimentation. Here are some observations you can make.

Observation 1: If the numbers are all positive, then the maximum number will decrease with each round.

Because the difference between two positive numbers is less than the larger of them, the maximum number must decrease.

Observations 2 - 4 are easier to try yourself than to read about why they are true.

Observation 2: If one of the numbers is 0 and the other two numbers are positive and different, the next round will have three positive numbers and the maximum number will be the same as the previous round.

Observation 3: If there are two 0's and a positive number, the next round will have that positive number twice and the third number will be 0.

Observation 4: If one of the numbers is 0 and the other two numbers are the same, all the rounds after that will be identical to that.

So, putting these observations together, the maximum number in a round will decrease if there are no 0's, and will stay the same for one round if there is a single or a double 0. That means that no matter what you start with, you will eventually reach a round with two or three 0's, and from then on it will not change.

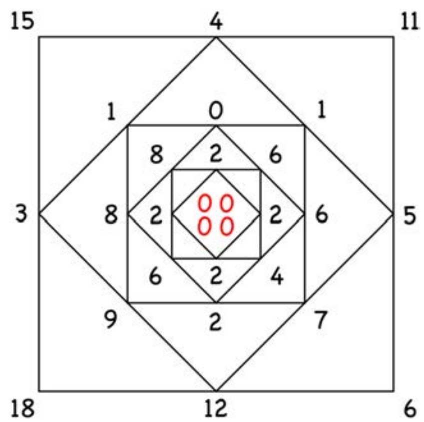
Observation 5: The only way to end up with three 0's is to start with three numbers that are the same and get to the three 0's in one round..

The first time three 0's occur, the previous round must have had three identical positive numbers. If you play with it briefly, you will see there is no way to produce three identical positive numbers by using differences from an earlier round.

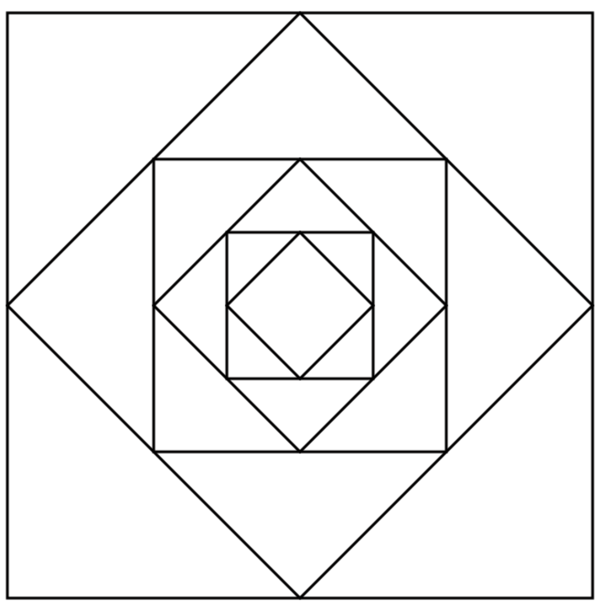
Puzzle of the Week

Difference Squares

Start by putting numbers in the corners of a square. Then, put the differences of those numbers in the middle of the sides and make a new square with those differences. Continue this process and see where it takes you.



THE CHALLENGE: 1) This seems to end with four 0's. If this is always true, can you explain why? 2) The example went to four 0's in the seventh round. Can you find examples that take much longer?



EXPLORATION: Investigate what happens if you use pentagons or hexagons instead.

Puzzle of the Week

Difference Squares – Notes

THE CHALLENGE: First, just play with it and see what happens starting with various groups of starting numbers. Look for patterns and enjoy the experimentation. Here are some observations you can make.

Observation 1: If the four numbers are all positive, then the maximum number will decrease with each round.

Because the difference between two positive numbers is less than the larger of them, the maximum number must decrease.

Observation 2: If there are one or more 0's, it may take more than one round, but after just a few rounds the maximum number will decrease (if it is not already 0).

This is just a matter of going through lots of cases of where the 0, or 0's, is relative to where the maximum number(s) is. They all work out, but it is tiresome to read about.

Because the maximum number must continue to decrease, eventually all the numbers must be 0.

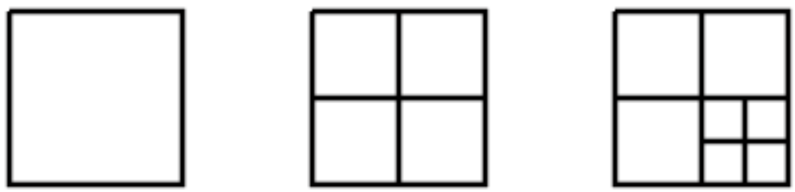
There is an alternative explanation that describes why all the numbers must end up being 0, but it uses more complex mathematics than you may want to read.

There is some sophisticated mathematics governing how to increase the number of rounds it takes to get to four 0's. For now, simply let your students enjoy playing while trying to improve their best scores.

EXPLORATION: If you are interested in learning more about this, a good place to start is to look up "Ducci Sequence" in Wikipedia.

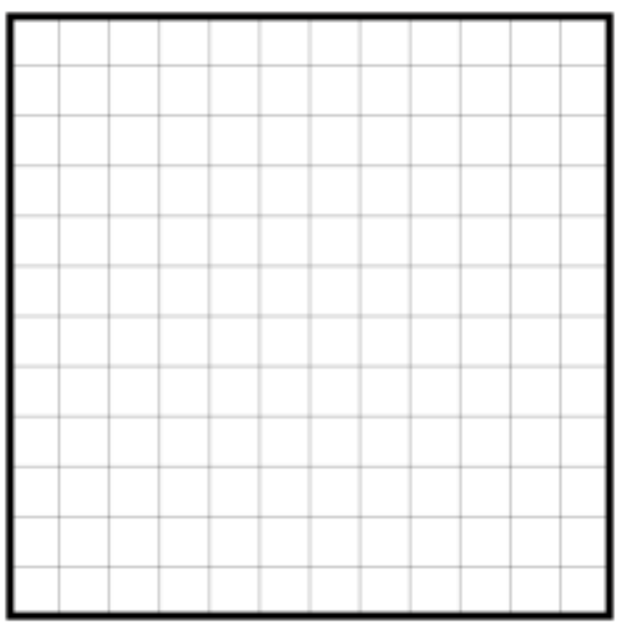
Puzzle of the Week
Filling Squares with Squares

Here is how to fill one large square with 1, 4, or 7 squares.



1 4 7

THE CHALLENGE: Find other square counts for filling a large square. Can you do it for 2, 3, 5, 6, 8, 9, or 10 squares?



EXPLORATION: When possible, find more than one way to get some of these numbers.

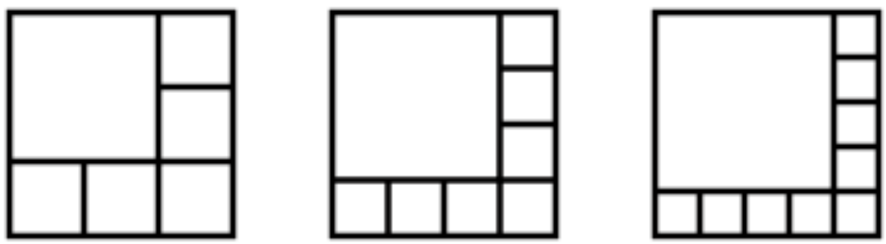
Puzzle of the Week

Filling Squares with Squares – Notes

THE CHALLENGE: Here is a systematic way of building up various counts.

Squares: A good place to start is with square numbers. It is easy to fill a square with 1, 4, and 9 squares by filling it, respectively, in a 1 by 1, 2 by 2, and 3 by 3 pattern.

Even Numbers: After some experimentation, you can create patterns for 6, 8, 10, or any other larger even number as follows. Start with a large square and then put smaller squares around two sides of it as in these drawings.



Replacing One Square: The next big step is to see that any one square in a solution can be replaced by any other existing solution. For example, this was done in producing the pattern for 7 in the introduction – the square in the bottom right corner for "4" was replaced with four smaller squares to produce "7.".

Whenever one square is replaced by four squares, that will increase the total square count by 3. Start with the list of solutions using square numbers and even numbers: 1, 4, 6, 8, 9, 10, 12, 14, and 16, and then add 3 to each entry on that list to get 4, 7, 9, 11, 12, 13, 15, and 17. Combining these two lists gives all the possibilities up through 17: 1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, and 17. Using these ideas, every number above 17 is easy enough, and so we reach the conclusion that:

Answer: Every number is possible except 2, 3, and 5.

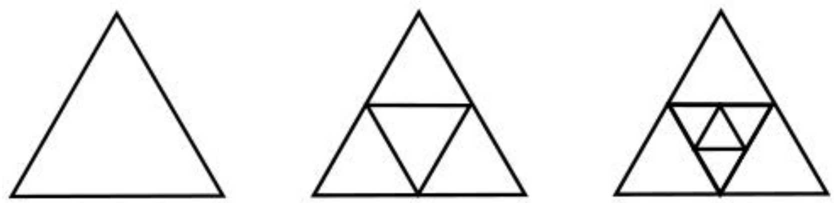
EXPLORATION: Some of these numbers can be produced in more than one way. For example, 9 can be done as a 3 by 3 pattern or as 6 plus three more.

There are many other patterns that can be created that are not made in the same way as these examples. For example, you can start with a 4 by 4 square and group 2 by 2 collections of squares into single 2 by 2 squares - each time you do that you will reduce the total count by 3.

Puzzle of the Week

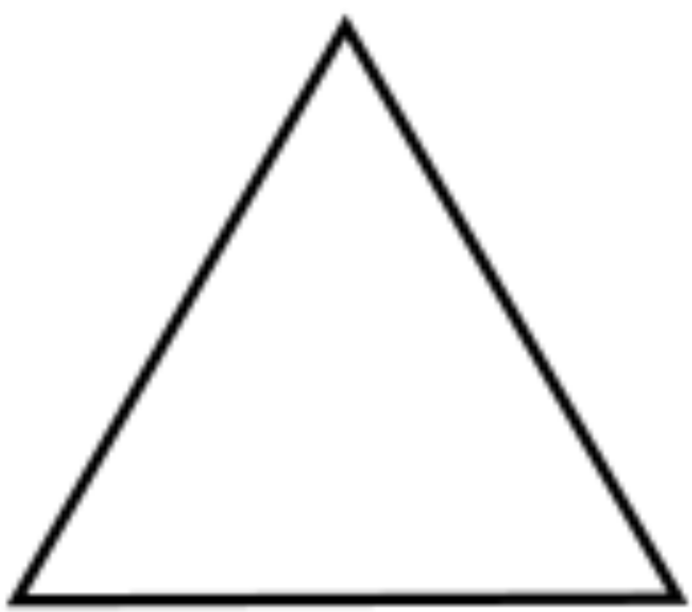
Filling Triangles with Triangles

Here’s how to fill one large triangle with 1, 4, or 7 triangles..



1 4 7

THE CHALLENGE: Find other triangle counts for filling a large triangle. Can you do it for 2, 3, 5, 6, 8, 9, or 10 triangles?



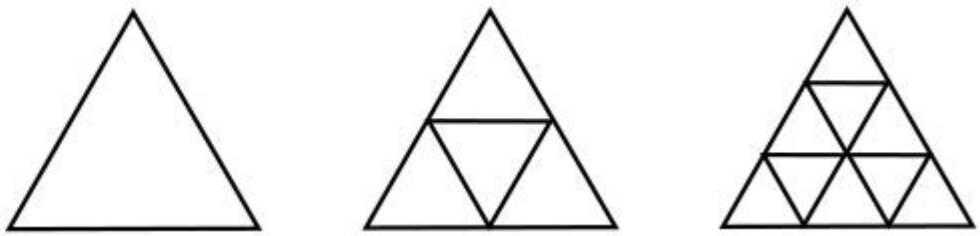
EXPLORATION: When possible, find more than one way to get some of these numbers.

Puzzle of the Week

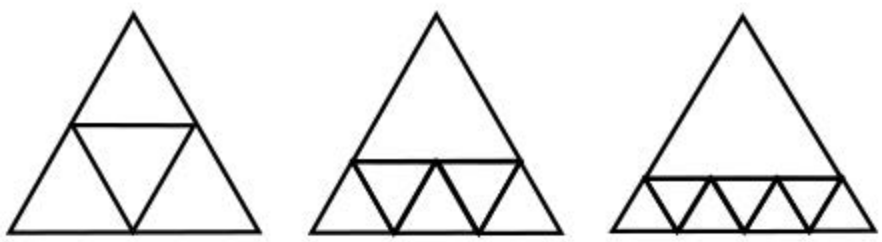
Filling Triangles w/ Triangles – Notes

THE CHALLENGE: Here is a systematic way of building up various counts.

Squares: A good place to start is with square numbers. It is easy to fill a triangle with 1, 4, and 9 triangles by filling it with the regular pattern used in the following illustration.



Even Numbers: After some experimentation, you can create patterns for 6, 8, 10, or any other larger even number as follows. Start with a large triangle and then put smaller triangles along one side.



Replacing One Triangle: The next big step is to see that any one triangle in a solution can be replaced by any other existing solution. For example, this was done in producing the pattern for 7 in the introduction – the triangle in the center for "4" was replaced with four smaller triangles to produce "7.".

Whenever one triangle is replaced by four triangles, that will increase the total triangle count by 3. Start with the list of solutions using square numbers and even numbers: 1, 4, 6, 8, 9, 10, 12, 14, and 16, and then add 3 to each entry on that list to get 4, 7, 9, 11, 12, 13, 15, and 17. Combining these two lists gives all the possibilities up through 17: 1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, and 17. Using these ideas, every number above 17 is easy enough, and so we reach the conclusion that:

Answer: Every number is possible except 2, 3, and 5.

EXPLORATION: Some of these numbers can be produced in more than one way. For example, 9 can be done as a 3 by 3 pattern or as 6 plus three more. Of course, there are other interesting ways to fill out a triangle to discover.

Puzzle of the Week

Letter Substitutions – 4

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$
\begin{array}{r} 8 \\ +\ A \\ \hline B\,2 \end{array}
\quad \Rightarrow \quad
\begin{array}{r} 8 \\ +\ 4 \\ \hline 1\,2 \end{array}
$$

THE CHALLENGE: Find the value of A, B, C, D, and E in these puzzles.

$$
\begin{array}{r} A \\ +\ B\,B \\ \hline C\,7 \end{array}
\qquad\qquad
\begin{array}{r} D \\ +\ E\,D \\ \hline D\,E \end{array}
$$

EXPLORATION: Make some letter substitution puzzles for your friends to solve.

Puzzle of the Week

Letter Substitutions – 4 – Notes

THE CHALLENGE: **A + BB = C7:** Adding A forces a carry, so C must be one more than B. A + B is 17, and the two numbers are one apart, so this must be 9 + 8 = 17.

The answer is 9 + 88 = 97.

D + ED = DE: Notice that D + D ends in E and has a carry. That is, D + D = 1E. Also, because of the carry to the tens place, D must be one more than E. List out the possibilities for D + D = 1E:

5 + 5 = 10, 6 + 6 = 12, 7 + 7 = 14, 8 + 8 = 16, 9 + 9 = 18.

The only one of these possibilities that fits the requirements is when D = 9 and E = 8.

The answer is: 9 + 89 = 98.

Puzzle of the Week

Letter Substitutions – 5

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$\begin{array}{r} 8 \\ +\ A \\ \hline B\ 2 \end{array} \quad \Rightarrow \quad \begin{array}{r} 8 \\ +\ 4 \\ \hline 1\ 2 \end{array}$$

THE CHALLENGE: Find the value of B, E, S, T, O, G, and U in these puzzles.

$$\begin{array}{r} B\ E \\ +\ B\ E \\ \hline S\ E\ E \end{array} \qquad \begin{array}{r} T\ O \\ +\ G\ O \\ \hline O\ U\ T \end{array}$$

EXPLORATION: Make some letter substitution puzzles for your friends to solve.

Puzzle of the Week

Letter Substitutions – 5 – Notes

THE CHALLENGE: **BE + BE = SEE:** For E + E to end in E, it must be that E = 0. The largest the carry can be from the tens place to the hundreds place is 1, so S = 1. Finally, B + B = 10 forces B = 5.

So, the answer is 50 + 50 = 100.

TO + GO = OUT: The largest the carry can be into the hundreds place is 1, so O = 1. O + O ends in T forces T = 2.

We now have 21 + G1 = 1U2. This means 2 + G = 1U. For 2 + G to be at least 10, G must be 8 or 9. If G = 9, then 2 + G = 11 and that would mean U = 1 and O = 1, which is not allowed. So, G = 8 and U = 0.

So, the answer is 21 + 81 = 102.

Puzzle of the Week

Letter Substitutions – 6

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$\begin{array}{r} 8 \\ +\ \ A \\ \hline B\ \ 2 \end{array} \quad \Rightarrow \quad \begin{array}{r} 8 \\ +\ \ 4 \\ \hline 1\ \ 2 \end{array}$$

THE CHALLENGE: Find the value of A, B, C, D, E, F, G, K, L, and M in these puzzles.

$$\begin{array}{r} B\ \ A \\ +\ \ B\ \ B \\ \hline C\ \ A\ \ B \end{array} \qquad \begin{array}{r} E\ \ D \\ +\ \ F\ \ D \\ \hline F\ \ G\ \ G \end{array} \qquad \begin{array}{r} K\ \ K \\ +\ \ L\ \ K \\ \hline L\ \ L\ \ M \end{array}$$

EXPLORATION: Make some letter substitution puzzles for your friends to solve.

Puzzle of the Week

Letter Substitutions – 6 – Notes

THE CHALLENGE: BA + BB = CAB**:** The carry can be at most 1, so C = 1. A + B = B forces A = 0. B + B = 10 finishes it off with B = 5.

The solution is: 50 + 55 = 105.

ED + FD = FGG: The carry must be 1, so F = 1. With F = 1, E + 1 + (possible carry) = 1G forces G = 0, and E = 8 or 9. D + D = 0 or 10 leaves us with D = 5, a carry into the next column, and then E = 8.

The solution is: 85 + 15 = 100.

KK + LK = LLM: Once again, the carry must be 1, so L = 1. K + 1 + (possible carry) = 11 forces K = 9 and the possible carry is a definite carry. Then K + K = 1M becomes 9 + 9 = 1M, so M = 8.

The solution is: 99 + 19 = 118.

Puzzle of the Week

Letter Substitutions – 7

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$\begin{array}{r} 8 \\ +\ A \\ \hline B\ 2 \end{array} \Rightarrow \begin{array}{r} 8 \\ +\ 4 \\ \hline 1\ 2 \end{array}$$

THE CHALLENGE: Find the value of A, B, C, D, E, F, K, L, and M in these puzzles.

$$\begin{array}{r} A\ A \\ +\ C\ B \\ \hline B\ B\ C \end{array} \qquad \begin{array}{r} D\ D \\ +\ D\ E \\ \hline F\ D\ F \end{array} \qquad \begin{array}{r} K\ K \\ +\ K\ K \\ \hline L\ L\ M \end{array}$$

EXPLORATION: Make some letter substitution puzzles for your friends to solve.

Puzzle of the Week

Letter Substitutions – 7 – Notes

THE CHALLENGE: **AA + CB = BBC:** The carry must be 1, so B = 1. A + B = C becomes A + 1 = C, so C is one more than A. A + C = 11 together with C = A + 1 forces A = 5 and C = 6.

The solution is: 55 + 61 = 116.

DD + DE = FDF: The carry must be 1, so F = 1. Looking through the possibilites, D + D + (possible carry) = 1D has only one way of working out - we must have D = 9 and the possible carry is a definite carry. Finally, D + E = 1F becomes 9 + E = 11 forces E = 2.

The solution is 99 + 92 = 191.

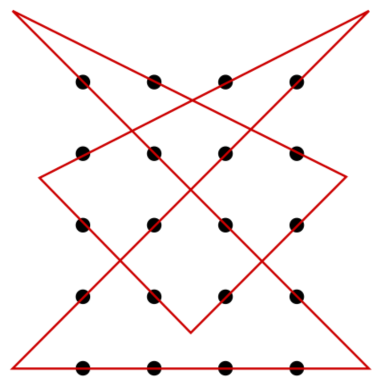
KK + KK = LLM: Again, the carry must be 1, so L = 1. For K + K + (possible carry) = 11, we must have K = 5 and the possible carry is a definite carry. 5 + 5 = 1M forces M = 0.

The solution is: 55 + 55 = 110.

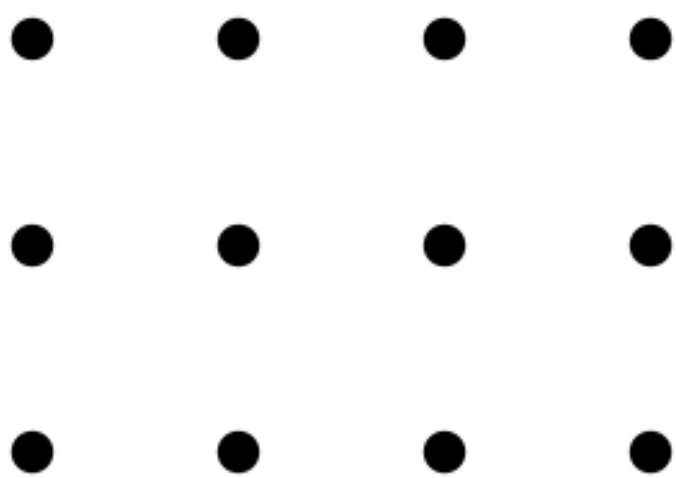
Puzzle of the Week

Lines – 6

Here is an example of a path created by 7 line segments drawn in one continuous drawing, without lifting the pen off the paper, that begins and ends at the same spot, and visits every dot on this 5 by 4 grid exactly once.



THE CHALLENGE: Find a continuous path using 5 line segments that begins and ends at the same point and that visits each of the points on this 3 by 4 grid exactly once.

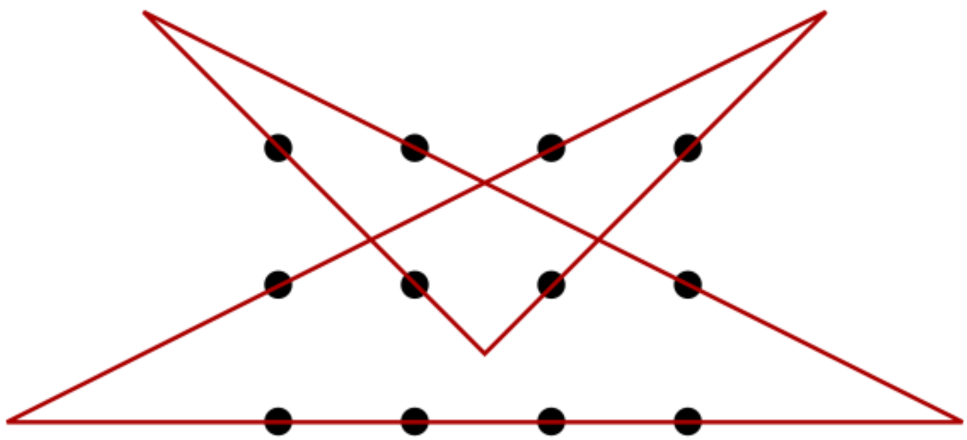


EXPLORATION: Experiment with larger grids and see how few line segments you can use.

Puzzle of the Week

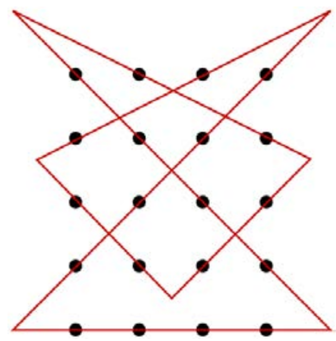
Lines – 6 – Notes

THE CHALLENGE: The answer is:

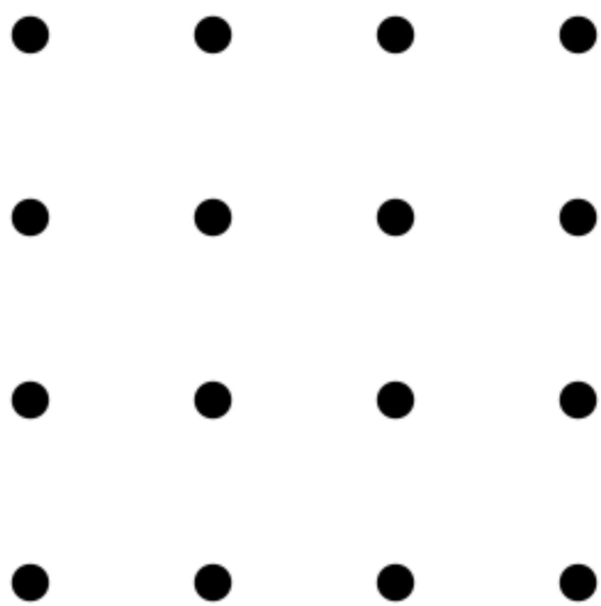


Puzzle of the Week
Lines – 7

Here is an example of a path created by 7 line segments drawn in one continuous drawing, without lifting the pen off the paper, that begins and ends at the same spot, and visits every dot on this 5 by 4 grid exactly once.



THE CHALLENGE: Find a continuous path using 6 line segments that begins and ends at the same point and that visits each of the points on this 4 by 4 grid exactly once.

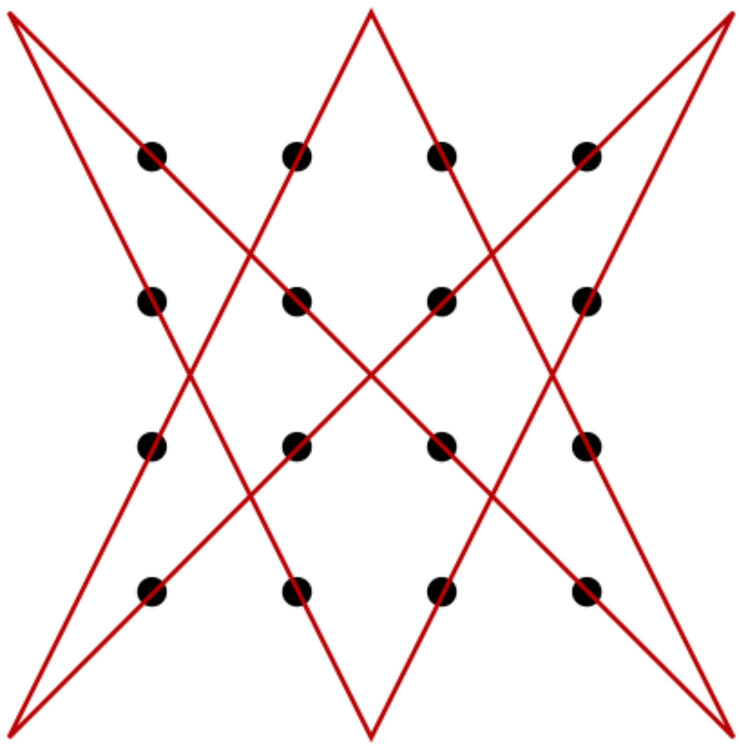


EXPLORATION: Experiment with larger grids and see how few line segments you can use.

Puzzle of the Week

Lines – 7 – Notes

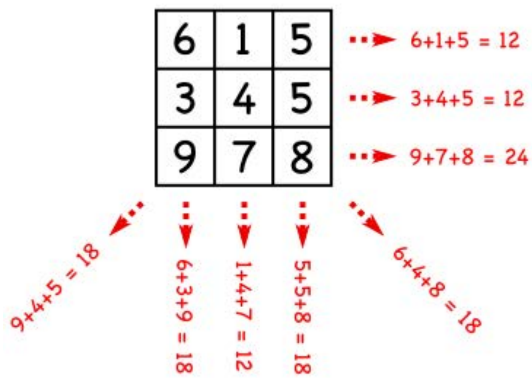
THE CHALLENGE: The answer is:

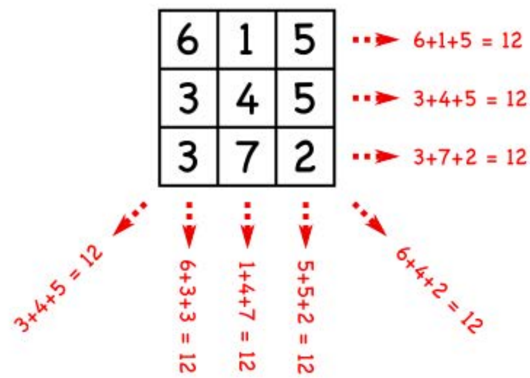


Puzzle of the Week

Magic Square – 5

In a ***Magic Square***, all the rows, columns and diagonals add up to the same number. This first square is <u>not</u> a Magic Square. The second one is a Magic Square with a constant sum of 12.





THE CHALLENGE: We have seen how to make a Magic Square with a set of numbers that are evenly spaced, such as {2, 6, 10, 14, 18, 22, 26, 30, 34}. Can you make a Magic Square that has no duplicate entries, where the numbers are not all evenly spaced?

EXPLORATION: Which shortcuts have you found for creating 3 by 3 Magic Squares? Can you devise a general method for constructing any Magic Square that has no duplicate entries?

Puzzle of the Week

Magic Square – 5 – Notes

THE CHALLENGE & EXPLORATION: As we have talked about in the earlier Magic Square puzzles, the central square must be the average of all nine numbers. Call that number “c.” The common sum will be 3c.

In the 1800’s, Ėdouard Lucas found a formula for generating all 3 by 3 Magic Squares that do not have repeated entries. If you pick two positive numbers a and b so that a < b < (c - a) and b is not 2a, then you have this Magic Square.

c–b	c+(a+b)	c–a
c+(b–a)	c	c–(b–a)
c+a	c–(a+b)	c+b

These numbers are, in increasing order: c - (a + b), c - b, c - (b - a), c - a, c, c + a, c + (b - a), c + b, c + (a + b).

For example, to get the numbers from 1 to 9, let c = 5, a = 1, and b = 3.

It’s interesting that the numbers larger than c exactly correspond to the numbers less than c.

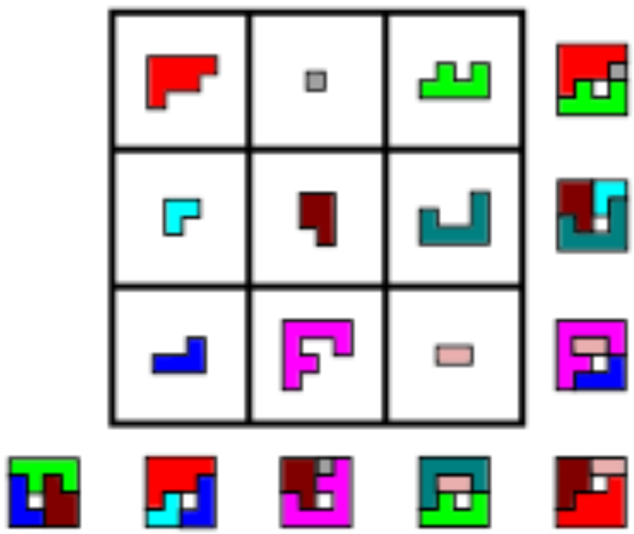
Armed with these formulas, simply pick a and b so that b - a is not 2a. For example, let a = 1 and b = 4. Let c = 6 to keep things from getting negative. With those values the numbers are: 1, 2, 3, 5, 6, 7, 9, 10, 11, and they produce this square with common sum 18.

2	11	5
9	6	3
7	1	10

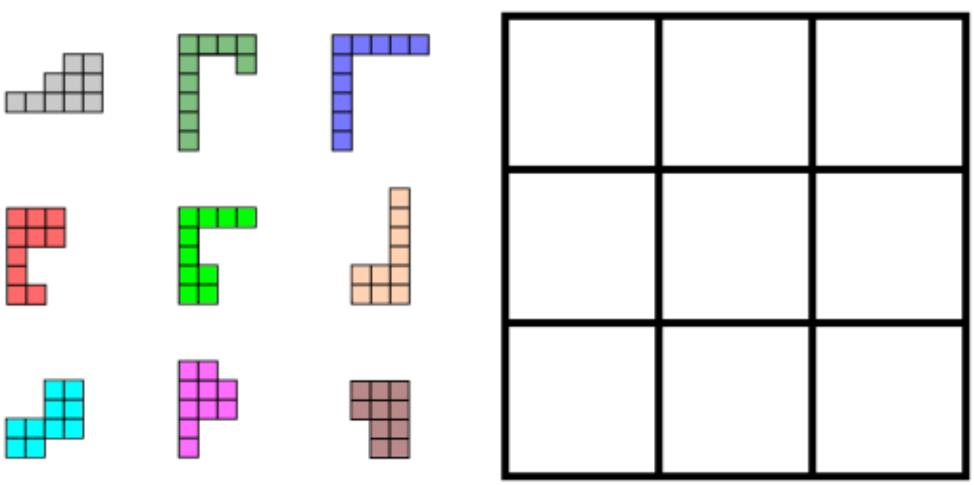
Puzzle of the Week

Magic Square – 6

In a ***Magic Square***, the numbers in the rows, columns and diagonals add up to the same number. In a ***Geometric Magic Square***, you "add" by putting together geometric shapes to form the same combined shape for each row, column, and diagonal. In this example, the pieces fit together (sometimes with rotations and flips) along rows, columns, and diagonals (as shown) to form the same 4 by 4 square with one 1 by 1 square missing.



THE CHALLENGE: Put the pieces in the square so the entries "add up" to a 5 by 6 rectangle in all directions.

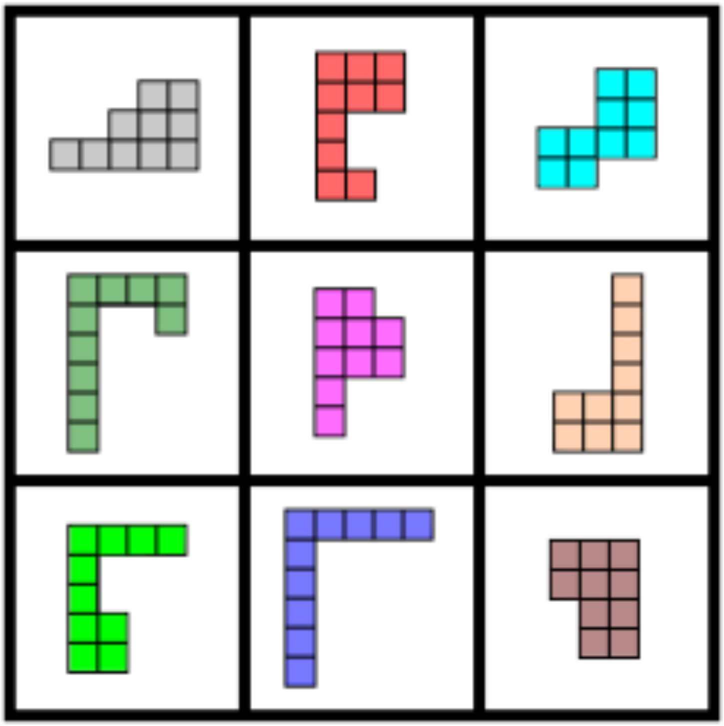


EXPLORATION: Play around with making your own Geometric Magic Squares.

Puzzle of the Week
Magic Square – 6 – Notes

THE CHALLENGE: There is a lot about Geometric Magic Squares online including on Wikipedia.

Here is the solution to this puzzle.



Puzzle of the Week
Pan Balance – 5

A pan balance tells you when its two sides are carrying the same amount of weight or whether one side is heavier than the other.

THE CHALLENGE: You have a 24-pound bag of flour. You need to measure out 9 pounds of the flour using a large pan balance which has no markings and no weights. How can you do it?

EXPLORATION: What other amounts of flour can you measure in this situation? Experiment with different starting amounts of flour and see what amounts you can create.

Puzzle of the Week

Pan Balance – 5 – Notes

THE CHALLENGE: In general, you can divide any amount you have into two piles of equal weight.

You can start by taking the 24 pounds of flour and pouring it out onto the pan balance until you have 12 pounds on each side. Put the two piles on the table. Take one of the 12-pound piles and split it into two equal piles of 6 pounds each. Put those two piles on the table. Take one of the 6 pound piles and split it into two equal piles of 3 pounds each. Put those two piles on the table.

At this point, you have four piles of flour that are 12 pounds, 6 pounds, 3 pounds, and 3 pounds. You can combine these in any way you like. In particular, you can make 9 pounds by combining the 6-pound pile with one of the 3-pound piles.

After reaching those last four piles, another way to create 9 pounds is to pour off 3 pounds from the 12-pound pile to balance with one of the 3 pound piles.

EXPLORATION: By combining 3, 6, and 12 in various ways, you can get the following pound amounts: 3, 6, 9, 12, 15, 18, 21, and 24.

Extrapolating from the example of starting with 24, it is clear we can start with any number, divide it by two as many times as possible and then create any amount that is a multiple of that smallest amount. For example, if we start with 20, we can divide it by two twice to get 5, and then be able to make any amount that is a multiple of 5.

Puzzle of the Week

Product Equals Sum

THE CHALLENGE: Suppose you have five positive whole numbers, A, B, C, D, and E, so that the following is true. What is the largest possible value of any of these numbers?

$$A \times B \times C \times D \times E = A + B + C + D + E$$

EXPLORATION: What would happen if you had a different number of numbers in this kind of equation?

Puzzle of the Week

Product Equals Sum – Notes

THE CHALLENGE: As you stare at the problem, you realize you want to get some leverage over the fact that, as the numbers increase, the sum will not grow nearly as fast as the product. One simple thing to do is to turn the sum into a product as follows.

Assume that the numbers, A, B, C, D, and E, are in decreasing order so that A is the largest value (it may be tied with others). Then A + B + C + D + E ≤ 5A. Consequently, A x B x C x D x E = A + B + C + D + E≤ 5A. Therefore, we have B x C x D x E ≤ 5. As these are whole numbers, that leaves us with very few possibilities to look at!

Each of these will be listed in B, C, D, E order.

- 5, 1, 1, 1: Then A x 5 x 1 x 1 x 1 = A + 5 + 1 + 1+ 1, so 5A = A + 8. This would mean A is 2, which can't happen because A is supposed to be the largest value.
- 4, 1, 1, 1: Then A x 4 x 1 x 1 x 1 = A + 4 + 1 + 1 + 1, so 4A = A + 7. This would mean 3A = 7, which is impossible.
- 3, 1, 1, 1: Then A x 3 x 1 x 1 x 1 = A + 3 + 1 + 1 + 1, so 3A = A + 6. This would mean 2A = 6, so A = 3. This gives the solution 3 x 3 x 1 x 1 x 1 = 3 + 3 + 1 + 1 + 1 = 9.
- 2, 1, 1, 1: Then A x 2 x 1 x 1 x 1 = A + 2 + 1 + 1 + 1, so 2A = A + 5. This would mean A = 5. This gives the solution that 5 x 2 x 1 x 1 x 1 = 5 + 2 + 1 + 1 + 1 = 10.
- 1, 1, 1, 1: Then A x 1 x 1 x 1 x 1 = A + 1 + 1 + 1 + 1, so A = A + 4, which is impossible.
- 2, 2, 1, 1: Then A x 2 x 2 x 1 x 1 = A + 2 + 2 + 1 + 1, so 4A = A + 6. This would mean A = 2. This gives the solution that 2 x 2 x 2 x 1 x 1 = 2 + 2 + 2 + 1 + 1 = 8.

Thus, there are three solutions: 3, 3, 1, 1, 1; 5, 2, 1, 1, 1; and 2, 2, 2, 1, 1. The largest number that appears in any of these is 5.

EXPLORATION: A very similar analysis can be done for any number of numbers. With larger numbers comes more to analyze.

For 2, 3, and 4 numbers we get:

- 2 x 2 = 2 + 2
- 3 x 2 x 1 = 3 + 2 + 1
- 4 x 2 x 1 x 1 = 4 + 2 + 1 + 1

Beyond these specific answers, after doing a number of examples it is "clear" that when there are n numbers, the largest number will always be "n" associated with the solution n x 2 x 1 x 1 x … x 1 = n + 2 + 1 + 1 + … + 1.

Puzzle of the Week

Removing Digits

THE CHALLENGE: Which ten digits would you remove (they don't need to be next to each other) from the number 1234512345123451234512345 to make the new number as large as possible? Which ten digits would you remove to make it as small as possible?

1234512345123451234512345

EXPLORATION: Can you think of other interesting problems similar to this one?

Puzzle of the Week
Removing Digits – Notes

THE CHALLENGE: To make the number as large as possible, the highest priority has to be putting 5's in the high order digits. So, start by removing 1 to 4 the first two times. Removing those 8 digits produces the number 55123451234512345. Removing the next 1 and 2 is the best you can do, and that produces the answer:

553451234512345

To make the smallest number, we want it to start with 1's. Removing 8 digits by removing the first two groups of 2 to 5 produces the number 11123451234512345. The best we can do is remove the next 4 and 5, and that produces the answer

111231234512345

EXPLORATION: Similar puzzles can be made using different sequences of digits. The theme for all of them is that the highest order digits are the most important.